An Introduction to Conformal Geometry

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Abstract

This term paper is meant to be an introduction to Conformal Geometry. Conformal Geometry is concerned with answering the following question: take a manifold with a pseudo-Riemannian metric. Now scale the metric by multiplying the metric with a positive function. You may imagine this as the manifold growing larger or smaller. What properties of the manifold, or differential operators on the manifold, remain unchanged, or are invariant? These invariants are known as conformal invariants, and constructing them helps us understand the conformal structure on these manifolds.

A handful of local invariants and invariant differential operators have been explored in this paper. As this paper is intended to only introduce the relevant notions, and perhaps give some intuition, no proofs have been provided. It is the author's belief that most of the statements in this paper can be proven using just definitions in a relatively straight-forward way.

An Introduction to Conformal Geometry

Introduction

Conformal geometry is the study of angle preserving transformations on a space. In two dimensions, conformal geometry is the study of Riemann surfaces; surfaces that locally look like the complex plane. In dimensions higher than two, conformal geometry is the study of Riemannian spaces or pseudo-Riemannian surfaces, with a class of metrics defined up to scale. This will be explained later, but basically means that scaling up the metric doesn't change the manifold. It is this equivalence of metrics upto scaling that converts a Riemannian manifold into a conformal manifold. Now we shall briefly discuss Riemannian manifolds, pseudo-Riemannian manifolds, and conformal manifolds, which are just manifolds with conformal structure.

Riemannian and pseudo-Riemannian manifolds

A Riemannian manifold M is defined such that for any $v, w \in T_pM$ for a point $p \in M$, $\langle v, w\rangle \geq 0$, and $\langle v, v\rangle = 0 \Longleftrightarrow v = 0$. A pseudo-Riemannian manifold M' is described such that for each non-zero vector $v \in T_pM$', there exists some vector w such that $\langle v, w\rangle \neq 0$. Note that this condition is much weaker than the existence of a positive definite inner product, as in a Riemannian metric.

Conformal manifolds

A conformal manifold is a pseudo-Riemannian manifold equipped with an equivalence class of metric tensors, in which two metrics g and h are equivalent if and only if $h = f^2g$ for some real valued smooth function f defined on the whole manifold. Note that f does not have to be a constant function. Hence, we can scale g differently at different points on the manifold to get h. Consequently, the equivalence class of metrics on conformal manifolds is huge.

The equivalence class of metrics defined above is known as a conformal metric or a conformal class. Hence, a conformal metric is defined only up to scale. A conformal metric is chosen by selecting any member of the conformal class, and then applying transformations on it that are "conformally invariant"; ie transformations that map a conformal class to another conformal class, making such a map well-defined.

A conformally flat metric is one in which the Riemann curvature tensor vanishes. A flat metric may be locally or globally flat. Clearly, global flatness is a much stronger condition than local flatness. THis is because it may only be possible to find a metric in the conformal class that is flat in a neighborhood of each point, but not flat globally. This is achieved by picking different members of the conformal class at each point, so that we can local flatness. But any individual member of the conformal class of metrics may not be globally flat. An n-sphere for instance is not globally flat, but is locally conformally flat. Euclidean space, on the other hand, with the Euclidean metric on it, is globally conformally flat.

Conformal Geometry

A conformal n-manifold $(n \geq 3)$ is the structure (M, c), where M is the n-manifold, and c is the conformal equivalence class of metrics of signature (p, q). The second point means that if f, g are two metrics in the same conformal equivalence class, then $f = e^{2\gamma} g$ for some $\gamma \in C^\infty(M)$.

To any pseudo-Riemannian manifold (M, g), we can associate a conformal manifold $(M, [g])$, where $[g]$ is the conformal equivalence class of g as described above. When we convert a Riemannian manifold into a conformal manifold, we are choosing to forget the notion of length, and only remember angles between lines. How is that? This is because of the existence of the equivalence class of metrics. Take $f, g \in [g]$, where $f = 2g$ for example.

Clearly, both of them measure the length of a given curve on the manifold differently (one measures it to be twice as long as the other). However, as they're in the same equivalence class, they *should* not measure a given property of a manifold, like the length of a curve on it, differently. Hence, length of a curve, and consequently distance between two points etc, cannot be meaningful properties to measure on a conformal manifold.

However, angles are a different matter. Metrics can measure angles between curves. And if two metrics are multiples of one another, they will still measure the same angle between two given curves. Hence, when we convert a Riemannian manifold (M, g) into a conformal manifold $(M, [g])$, the angles between curves remain the same, although the lengths of curves are defined in the Riemannian manifold although they are not defined in the conformal manifold.

For example, imagine a Lorentzian conformal manifold of signature $(n-1, 1)$. According to mathematical convention, the number of coordinates with positive eigenvalues is given first, which is $n-1$ in number, and the number of coordinates with negative eigenvalues is given second. As lengths are no longer meaningful in this conformal manifold, the spacetime length of a worldline, or the spacetime distance between two points (called "events" in Relativity) has no meaning anymore. However, the angles between two worldlines is still well-defined for all metrics in $[g]$. Hence, only the light cone structure of original Lorentzian manifold is retained. What does this mean?

A light cone is a cone-like structure at each point (event), that separates events that can happen starting from the node of the cone (time-like events) from events that cannot happen if they were originally the event at the node of the cone(space-like events). Time like events are all those events that are at certain angles from the walls of the light cone, and similarly space-like events are all those events that are at certain angles from the walls of the light cone. As angles are preserved while converting the pseudo-Riemmanian Lorentzian manifold to a conformal manifold, the light cone structure, or the structure separating time-like and space-like events at each point, is preserved.

The significance of conformal geometry for relativity stems from the fact that we mainly only care about the causal structure in relativity (what events can happen, and what events cannot). As conformal geometry captures that information while leaving out other irrelevant information, the conformal structure seems like a useful structure to have on spacetime. Moreover, the metric may also evolve with time, and we want to study the properties of spacetime that remain invariant with time.

Torsion tensor and Curvature

We are going to assume here that a reader is familiar with what a tensor and a connection is. For a connection ∇, a torsion tensor T^∇ is a global section in $\Gamma(TM \otimes \wedge^2 T^* M)$, which means that it is a $\binom{1}{2}$ tensor. For any two vector fields u, v, it is defined as

$$T^\nabla(u, v) = \nabla_u v - \nabla_v u - [u, v]$$

This is still a $\binom{1}{0}$ tensor, and is hence a vector. It needs to be verified that this is indeed a tensor, which means it is multi-linear over $C^\infty(M)$. Also, by the construction, it is clear that the torsion tensor is coordinate independent, as each term $\nabla_u v$, $\nabla_v u$ and $[u, v]$ individually is coordinate independent.

For any connection ∇ on a manifold, the curvature is defined as

$$R^\nabla(u, v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]} w$$

Here $u, v, w \in \Gamma(TM)$. Clearly, this is also coordinate independent.

The torsion tensor has been defined because we will mainly be dealing with the Levi-Civita connection in this paper, which is a connection with 0 torsion tensor. The curvature tensor has been defined because will be extensively used in the paper.

Ricci Calculus and Weyl's Invariant Theory

Recall that a pseudo-Riemannian manifold has a metric g of signature (p, q) defined on it. Each metric can be uniquely mapped to a Levi-Civita connection ∇_g on the manifold. A Levi-Civita connection is defined as a connection which has 0 torsion tensor, and also the property that $\nabla_g g = 0$. The way that this mapping works is: an infinite number of connections can determine the same metric. This is intuitively clear, as bending a manifold in an ambient (perhaps Euclidean) space may change the connection, but not the metric. Amongst these infinite connections, there is a torsion-free connection too. Map the metric to this torsion-free connection ∇_g.

Now map this torsion-free connection to the Riemannian curvature R^g, which is the curvature determined by a torsion-free connection. Composing these two maps, we get the map $g \to R^g$. We can say that R^g is an invariant of the pseudo-Riemannian manifold (M, g), as however we deform the manifold, as g is fixed, it will map only to R^g through the aforementioned map. How can we construct more such invariants?

As g, ∇_g and R^g are invariants of (M, g), we can construct more such invariants by combining them using multiplication, contraction, etc. If the Riemannian curvature is denoted as $R_{ab}{}^{c}{}_{d}$, then some invariants are Ricci curvature $R_{ab} = R_{ca}{}^{c}{}_{b}$, scalar curvature $Sc = g^{ab}R_{ab}$, other invariants like $\nabla_a R_{bc}{}^{d}{}_{e}$, etc. Let us call this set of invariants, generated by combinations of g, ∇_g and R^g, Weyl invariants. The question we have to ask is do all **local** Riemannian invariants arise this way? Are all local Riemannian invariants Weyl Invariants?

First, we need to study what a local invariant is. There are many classes of local invariants, one of which is scalar invariants. A scalar invariant $P(g)$ is an invariant that is (1) *natural*, in the sense that for any diffeomorphism $\phi : M \to M$, we have $P(\phi^* g) = \phi^* P(g)$, and (2) is a polynomial in g_{ab} and its partial derivatives with respect to local coordinates.

It is known that all scalar invariants are indeed Weyl invariants, and this theorem is known as the ***Weyl's Classical Invariant Theory***. This is one of the rare instances in

Mathematics where we have a complete classification of a concept.

However, what about linear invariant operators, as opposed to just scalar invariants? This is a much more delicate issue, and will be dealt with later.

Conformal Transformations and Conformal Covariance

We want some way of being able to construct all local invariants and all invariant linear differential operators in a conformal structure.

Consider a metric $g \in [g]$, where $[g]$ is the conformal class of g. Then there exists another metric $\hat{g} = f^2 g$ in the same conformal class. How is the Levi-Civita connection ∇^g, corresponding to the metric g, different from $\nabla^{\hat{g}}$ corresponding to the metric $\hat{g}$?

Consider the operator $\gamma_a = f^{-1}\nabla^g_a f$. Then for a vector field v^a, we have

$$\nabla^{\hat{g}}_a v^b = \nabla^g_a v^b + \gamma_a v^b - \gamma^b v_a + \gamma^c v_c \delta^b_a$$

Similarly, for a covector ω_b, we have

$$\nabla^{\hat{g}}_a \omega_b = \nabla^g_a \omega_b - \gamma_a \omega_b - \gamma_b \omega_a + \gamma^c \omega_c g_{ab}$$

From the second formula, we can conclude that the map $\omega \to \nabla_a \omega_b - \nabla_b \omega_a$ is invariant. In other words,

$$\nabla^g_a \omega_b - \nabla^g_b \omega_a = \nabla^{\hat{g}}_a \omega_b - \nabla^{\hat{g}}_b \omega_a$$

which means $\nabla_a \omega_b - \nabla_b \omega_a$ doesn't change when the metric changes from g to $\hat{g}$.

We can in fact extend this analysis to $\binom{0}{2}$ tensors. Say F_{bc} is such a tensor. Then $\nabla_{[a}F_{bc]}$ is conformally invariant. Hence, so is the map $F_{bc} \to \nabla_{[a}F_{bc]}$.

We can observe a general pattern here. The Levi-Civita connection clearly changes with the change in metric, and hence so does its action on vector fields, co-vector fields and other tensors. However, because we can explicitly write down this change, we can form combinations of the connection and tensors which remain invariant. This seems like a direct and obvious way of constructing invariants. We will later explore how this approach fails while trying to construct invariant linear differential operators.

Now we shall talk a little about conformal covariance instead of conformal invariance. Conformal covariance means that if the metric changes, an operator in the original metric gets multiplied by a positive function in the new metric. Consider a skew-symmetric $\binom{0}{2}$ tensor F_{bc}. Then

$$\hat{\nabla}^b F_{bc} = f^{-2}(\nabla^b F_{bc} + (n-4)\gamma^d F_{dc})$$

Quite clearly, if $n = 4$, then $\hat{\nabla}^b F_{bc} = f^{-2}(\nabla^b F_{bc})$, which means that $\nabla^b F_{bc}$ is conformally covariant in 4 dimensions. Similarly, in 4 dimensions, $\nabla^b \nabla_{[b} u_{c]}$ is also conformally covariant. Under certain assumptions, these statements about conformal invariance are a re-statement of Maxwell's equations of Electromagnetism.

Conformal rescaling and Curvature

Using the conformal transformation formulae for vectors and co-vectors, we should be able to compute the conformal transformation formulae for Riemannian curvature and its derivatives. This seems to be a promising way of finding all conformal invariants, at least at the very lowest orders.

Let us fix a metric g. We can decompose the curvature tensor of g in the following way

$$R_{abcd} = \underbrace{W_{abcd}}_{\text{trace free}} + \underbrace{2g_{c[a}P_{b]d} + 2g_{d[b}P_{a]c}}_{\text{trace part}}$$

Here $W_{ab}{}^{c}{}_{d}$ is known as the Weyl tensor, and P_{bc} is known as the Schouten tensor. One can see that $W^{g}{}_{ab}{}^{c}{}_{d} = W^{\hat{g}}{}_{ab}{}^{c}{}_{d}$. Hence, the Weyl tensor is a conformal invariant. It is important to note that the Schouten tensor is not conformally invariant, as $P^{\hat{g}}{}_{ab} = P_{ab} - \nabla_a \gamma_b + \gamma_a \gamma_b - \frac{1}{2} g_{ab} \gamma_c \gamma^c$.

Something that is conformally covariant but not invariant is $W_{abcd}W^{abcd} = |W|^2$. One can prove that $|\hat{W}|^2 = f^{-4}|W|^2$. Hence, the curvature tensor in itself is not conformally invariant, although it can be decomposed as the sum of two tensors of which one is conformally invariant.

Conformally Invariant Linear Differential Operators

We want to construct conformally invariant linear differential operators $D^g : \mathcal{U} \to \mathcal{V}$ such that $D^{\hat{g}} = D^g$. Our general approach until now would suggest the following algorithm: calculate how an operator changes on conformal re-scaling. Then find some linear combination of this operator with perhaps other operators such that conformal re-scaling leaves it invariant. It turns out that this is pretty difficult to do!

Consider the Laplacian $\Delta = \nabla^a \nabla_a$. Then for any smooth function h, one can calculate that

$$\Delta^{\hat{g}} h = f^{-2}(\Delta^g h + (n-2)\gamma^c \nabla^g_c h)$$

Clearly, Δ is conformally covariant (forget invariant) only for $n = 2$.

We can however modify the Laplacian to construct a conformally covariant differential operator. The modified Laplacian is known as the conformally invariant laplacian, and is given by

$$Y^g = \Delta^g + \frac{n-2}{4(n-1)} Sc^g$$

Here Sc denotes the scalar curvature.

Conclusion and Future Directions

In this paper, we have explored how some tensors and other operators change when the metric on a pseudo-Riemannian manifold is conformally re-scaled. Operators like connection and curvature don't change "nicely". However, in some situations we can find clever combinations of operators, or perhaps parts of those operators, that do change "nicely". This paper is very limited in scope, mostly due to the author's lack of expertise in the field, and the limited applicability of the technique used to construct most of these invariants. As has been demonstrated above, the technique of finding clever combinations of ∇^g and other operators that remain invariant or covariant doesn't lead us very far. However, Charlie Fefferman has used the technique of constructing the Ambient metric, finding invariants in a higher dimensional pseudo-Riemannian manifold, and then using that information to construct invariants in the original pseudo-Riemannian manifold, with great success. I hope to study this technique for my PhD, and hopefully shed more light on it in the future.

References

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