Topology Quals Preparation

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1. **Show that $O(n)/O(n-1) \cong S^{n-1}$, $SO(2) \cong S^1$.**

Consider the group action of $O(n)$ on the point $p = (1, 0, \ldots, 0, 0) \in S^{n-1}$. As the action of an orthogonal matrix preserves norm, $O(n)$ maps p to other points in $O(n)$. Moreover, this action is transitive. This is because for any othe point q in S^{n-1}, let $A \in O(n)$ such that the first column of A is q and the other columns are any other columns as long as orthonormality is satisfied (consider an orthonormal basis with q as one of its elements. Such matrices definitely exist).

By the orbit stabilitizer theorem, we know that there exists an injective and surjective (bijective) between $O(n)/H$ and S^{n-1}, wehre H is the stabilizer of the group action of $O(n)$ on p. We shall now prove that H is $O(n-1)$. Consider the following embedding of $O(n-1)$ into $O(n)$: if $(a_{i,j})$ is the matrix in $O(n-1)$, map each element $a_{i,j}$ to a matrix where it is the $i+1, j+1$ element. In the $1, 1$ position, put 1, and put 0 in the other entries of the first row and first column. Clearly this is an element of $O(n)$. We will now prove that the image of $O(n-1)$ is the set of stabilizers of ths group action described above. Clearly, $O(n-1)$ is contained within the stabilizer. Now let a be an element of the stabilizer. The first column of a has to be p itself. Because each row of an orthonormal matrix has to have norm 1, and 1 is already there in the $1, 1$ position, the rest of the elements of the first row also have to be 0's. As for the remaining $(n-1) \times (n-1)$ matrix, it should contain rows of norm 1, and be orthonormal to each other. This corresponds to exactly the embedding of $O(n-1)$ into $O(n)$. Hence, the stabilizer is exactly $O(n-1)$.

Now note that $O(n)$ has a continuous group action on $(1, 0, \ldots, 0)$. Moreover, $O(n)/O(n-1)$ is compact as $O(n)$ is compact. Hence, we have a continuous bijection from a compact space. This proves that $O(n)/O(n-1) \cong S^{n-1}$.

We know that $O(2)/O(1) \cong S^1$. Consider the group action of $SO(2)$ on the point $(1, 0)$ in S^1. As before, it is transitive. Now let's compute the stabilizer of this group action. As shown before, the first row and column are just $(1, 0)$. The remaining element in the matrix, which is the $(2, 2)$ element, has to be 1 because the determinant of the matrix is 1. Hence, as the stabilizer is I, we have the isomorphism $SO(2) \cong S^1$.

Also, topologically, we have a coninuous bijection from $SO(2)$ to S^1. Hence, $SO(2) \cong S^1$.

2. **If $X \neq \emptyset$, arc wise connected the augmentation $\epsilon : H_0(X) \to Z$ is an isomorphism.**

As the space is arcwise connected, $H_0(X)$ contains only elements of the form $n[p]$, where $n \in Z$ and $[p]$ is the equivalence class of points in X. We have $np \sim n'$ for any $p' \in X$. Hence, as $H_0(X)$ contains only elements of the form $n[p]$, and ϵ takes np to n, $H_0(X) \cong Z$.

3. **What are $H_i(D^n, S^{n-1})$, $H_i(S^n, D^n_+)$ where D^+_n is the north hemisphere?**

We first write down the exact sequence $0 \to S^{n-1} \to D^n \to D^n/S^{n-1} \to 0$. This gives us the complex $\cdots \to H_p(S^{n-1}) \to H_p(D_n) \to H_p(D^n, S^{n-1}) \to H_{p-1}(S^{n-2}) \to H_{p-1}(D^{n-1}) \to \ldots$. This tells us that $H_p(D_n, S^{n-1}) = H_{p-1}(S^{n-2})$.

We shall again use relative homology for this. We get $H_i(S^n, D^n_+) = H_{i-1}(D^+_n) = 0$.

4. **Prove that S^{n-1} is not a retract of D^n.**

We shall prove this by contradiction. Assume that there does exist such a retract- let us call it ϕ. Consider the following maps $S^{n-1} \xrightarrow{id} D^n \xrightarrow{\phi} S^{n-1}$. Then $\phi \circ id$ is the identity map on S^{n-1}, which should induce the identity map on $\pi_{n-1}(S^{n-1})$. However, this is impossible as $\pi_{n-1}(D^n) = 0$. Hence, there does not exist a retract of D^n to S^{n-1}.

5. **Find several CW structures on S^2, RP^2, T^2**

Answer- Refer to Hatcher.

6. **Define the CW-homology and explain the differential.**

Read the description given on wikipedia.

7. **If $p : (E, y_0) \to (X, x_0)$ is a covering map, show that $\pi_n(E, x_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$. Note: $\pi_n(E, x_0)$=nth homotopy group**

Higher homotopy groups π_n are homotopy equivalence classes of maps of S^n to a manifold. Let $a : S^n \to (X, x_0)$ be a map. We know that it can be lifted to (E, y_0). However, as we're lifting paths from S^n and not I^n, the lifts will also be maps of S^n based at y_0. Moreover, if $[a] \sim a'$, then their lifts are also homotopic. Conversely, given maps from S^n to (E, y_0), we have maps from S^n to (X, x_0). Moreover, projection preserves homotopy. Clearly the map from $\pi_n(X, x_0)$ to $\pi_n(E, y_0)$ is surjective. Moreover, it is also injective as if the lifts of two maps are homotopic in E, then composing those lifts with projection gives a homotopy between the original paths. Hence, we have an isomorphism between $\pi_n(E, y_0)$ and $\pi_n(X, x_0)$, which proves the claim.

8. **Prove that if G is a finite group acting freely on a Hausdorff space X, then the action of G on X is properly discontinuous.**

A group action is considered free if the only element which fixes any point in the topological space is e. In other words, if $gx = x$ for any $x \in X, g \in G$, then $g = e$.

Moreover, the group action $x \to gx$ is continuous for each $g \in G$. This means that the inverse of any open set is open.

For a $g \in G$, let n be the order of g. For $x \in X$, take disjoint neighbourhoods around x and gx. Let these neighbourhoods be U, V respectively. Then the inverse of V is an open set containing x. consider $g^{-1}(V) \cap U$. Then this is an open set containing x that maps to within V, and hence is disjoint from its image within V. This proves that the action of G on X is properly discontinuous.

9. **Let $p : \tilde{X} \to X$ be a covering space, where $\tilde{X}$ is simply connected. Show that there is a bijection between the sets $\pi_1(X, a)$ and $p^{-1}(p(a))$ for all $a \in \tilde{X}$**

If $f \sim g$ in X, and the unique lifts of f, g are $\tilde{f}, \tilde{g}$, then $\tilde{f} \sim \tilde{g}$. Assuming that we always lift paths starting at a point $a \in \tilde{X}$, we see that homotopic paths always have the same end points in the fiber $p^{-1}(a)$. Hence, we have a well-defined map $\pi_1(X, p(a)) \to p^{-1}(a)$. This map is injective because if two two paths have the same end point, then as the covering space is simply connected, the two paths are homotopic. This map is also surjective because any path from a to any other point in the fiber (say a') maps to a path in $\pi_1(X, p(a))$, and hence lifts to this path from a to a'. Hence we have the bijection we need.//

10. **Prove that $\pi_1(S^n) = 0$, for $n \geq 2$.**

The basic motivation for the proof will be the following: if we can prove that any map $f : (I, \partial I) \to S^n$ can be homotopically mapped to a path that does not contain at least one point in S^n for $n \geq 2$, we'll be done. This is because $S^n\{a\}$ for $a \in S^n$ is homeomorphic to R^n. Hence, we can homeomorphically map the path to R^n, homotopically deform it to the null map at its base point, and then pull back the homotopy to give us a null homotopy of the original path. We shall now set out to prove what we want, as explained above.

For a path $f : (I, \partial I) \to S^n$, consider any point a that is not the base point of f, and construct an open disc $N(a)$ whose closure does not contain the base point x_0. $f^{-1}(N(a))$ is an open set in I and hence is a union of disjoint open intervals $\{\cup(a_i, b_i)\}$. Each of these intervals is mapped to to portions of the path f contained within $N(a)$. Moreover, $f(a_i), f(b_i)$ lie at the boundary of $N(a)$. This is because the path f does not start at or end at any point in $N(a)$. Hence, $f((a_i, b_i))$ is contained within a connected segment of f whose endpoints are at the boundary of $N(a)$. As the pre-image of this whole component is contained within $\{\cup(a_i, b_i)\}$, if $f(a_i)$ and $f(b_i)$ are not the end points of this segment, this would imply that the connected line segment in $N(a)$ is not connected, which is a contradicton. Hence, $f(a_i), f(b_i)$ lie at the boundary for all i.

Now consider $f^{-1}(a)$. This is a compact set as it is a closed set within I. Now just consider all (a_i, b_i) that contain $f^{-1}(a)$. As $f^{-1}(a)$ is compact, we can find a compact subcover $\cup_{i=1}^{n}(a_i, b_i)$ of $f^{-1}(a)$. Hence, we now understand that only finitely many line segments of f contained within $N(a)$ pass through a. We can now deform these finitely many line segments connecting $f(a_i)$ and $f(b_i)$ to segments that lie along the boundary of $N(a)$, and connect $f(a_i)$ and $f(b_i)$. We can do this because of the convexity of S^n. We needed finitely many such segment because only then can be combine all these individual homotopies to construct a homotopy from f to a path that does not contain a. This is because the final homotopy obtained by combining all the individual homotopies is constructed inductively.

As stated above, we can now deform the path to a path not containing a, and hence to the null path at x_0. This proves that $\pi_1(S^n) = 0$.

11. **State (do not prove) van Kampen's Theorem. Use it to give $\pi_1(\infty)$.**

Let U_1, U_2 be path connected spaces such that $U_1 \cup U_2$ be path connected.
Version 1: If $U_1 \cap U_2$ is simply connected, then $\pi_1(U_1 \cup U_2) = \pi(U_1) * \pi_1(U_2)$

By a direct application of this theorem, we see that $\pi_1(\infty) = Z * Z$

12. **Does there exist a topological space Y such that $S^1 \times Y$ is homeomorphic to RP^2 or S^2?**

This is impossible as $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

13. **Prove that a compact subset of a Hausdorff space is closed.**

Trivial

14. **$f : X \to Y$ continuous from compact X to Hausdorff Y. Show**

that f is a homeomorphism iff it is a bijection.
Follows from previous question

15. **Suppose f, g, h are three paths in X with $f(1) = g(0)$ and $g(1) = h(0)$. Prove that $(f * g) * h = f * (g * h)$**

Too long to explain. JP May's book has illustrative diagrams.

16. **Prove that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$**

We have the following map $\pi_1(X) \times \pi_1(Y) \to \pi_1(X \times Y)$, which maps $[p(t)], [q(t)]$ to $[p(t), q(t)]$. This map is well defined because if F is the homotopy between p and p', and G is the homotopy between q, q', then F, G is the homotopy between $[p, q]$ and $[p', q']$.
This map is injective as if $[p, q]$ is nullhomotopic, which means there exists a homotopy H from $[p, q]$ to $[0]$, then $p_1 \circ H$ is a homotopy from $[p]$ to 0, and $p_2 \circ H$ is a homotopy from q to 0. Hence, both $[p]$ and $[q]$ are 0.Here p_1 and p_2 are projection maps
This map is surjective because any element $[a]$ in $\pi_1(X \times Y, (x_0, y_0))$ is the image of the element $(p_1[a], p_2[a])$ in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.
Hence, we have a group isomorphism.

17. **Prove that $\pi_1(G, e)$ is abelian for G a topological group. Deduce that $\pi_1(G, e) \cong H_1(G)$**

Consider the paths $a, b : I \to G$. We need to prove that $[a * b] = [b * a]$.

Consider the map $f : I^2 \to G$ such that (s, t) is mapped to $a(s).b(t)$. As G is a topological group, this map is defined. We see that both $a * b$, which is the image of $(0, 0) \to (1, 0) \to (1, 1)$ under f, and $b * a$, which is the image of $(0, 0) \to (0, 1) \to (1, 1)$ under f, are homotopic to the path $a(t).b(t)$. Hence, $[a * b] = [b * a]$.This proves that the fundamental group of a topological space is abelian.

Hurwitz Theorem says that $\pi(G, e)/[\pi(G, e), \pi(G, e)] \cong H^1(G)$.
As $[\pi(G, e), \pi(G, e)] = 0$, we have $\pi_1(G, e) \cong H^1(G)$.

18. **Prove that $\pi_1(S^1) \cong Z$, $\pi_1(RP^2) = Z_2$**

Proving that $\pi_1(S^1) = Z$ is a standard exercise. Now we shall use this fact to prove the second assertion.
We know that S^1 is a two-cover of RP^2, which means each fiber contains two points. This covering space is simply connected. We have a theorem saying that the fundamental group of the base space has the same cardinality as the number of fiber points, if the space is simply connected. Also, any group of cardinality 2 is isomorphic to Z_2. Hence, $\pi_1(RP^2) = Z_2$.

19. **Sketch the proof that if f is homotopic to g, $f, g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$**

This proof involves constructing maps $h_n : C_n(X) \to C_{n+1}(Y)$, and then showing that $f - h : C_n(X) \to C_n(Y) = \partial \circ h_n + h_{n-1} \circ \partial$, which implies that $f - g : H_n(X) \to H(Y)$. The proof in full is too long to write in an exam.

20. **Compute $H_i(S^n)$ using the Mayer-Vietoris sequence.**

We can construct the following complex: $\dots \to H_i(S^{n-1}) \to H_i(D^n) \oplus H_i(D^n) \to H_i(S^n) \to H_{i-1}(S^{n-1}) \to H_{i-1}(D^n) \oplus H_{i-1}(D^n) \to \dots$.

As D^n is collapsible, we can conclude that $H_i(S^n) = H_{i-1}(S^{n-1})$

21. **Show that any continuous map $f : D^n \to D^n$ has a fixed point (Brower).**

Let us suppose that f has no fixed point. Consider the ray starting from $f(p)$, passing through p, and hitting the boundary S^1. This ray is well defined as $p \neq f(p)$. Also, for each point on the boundary, the ray starts at $f(p)$ and ends at p itself. Hence, this is a continuous retract of D^n to S^{n-1}. We know that this is impossible by Question 4.

22. ***Prove that if $f : S^n \to S^n$ has no fixed point, then f is homotopic to the antipode map, and it has index $(-1)^{n+1}$.***

If f does not have a fixed point, consider the homotopy $\frac{(1-t)f(p)+t(-p)}{|(1-t)f(p)+t(-p)|}$. The denominator can never be 0 as that would imply that $f(p) = \frac{t}{1-t}p$. We know that in S^n, two points can be scalar multplies of each other only if the scalar is ± 1. Here 1 is impossible. For $\frac{t}{1-t} = -1$, we get a nonsensical solution. Hence, we have a well-defined homotopy between f and the antipodal map.

The degree of the antipodal map is $(-1)^{n+1}$ because it can be thought of as the composition of $n+1$ reflections about the $n+1$ axes in the R^{n+1} space that S^n can be thought of as embedded in.

23. ***If $g : S^n \to S^n$ is nullhomotopic $\implies$ g has a fixed point.***

A mail that does not have a fixed point is homotopic to the antipodal map, and hence has degree $(-1)^{n+1}$. A map that does not map any point to its antipode is homotopic to the identity, and hence has degree 1. Hence, a nullhomotopic map, which has degree 0, both maps a point to itself, and maps another point to its antipode. Hence proved.

24. **$f : S^2 \to R^2$ continuous, there exists $x \in S^2$ such that $f(x) =$**

$f(-x)$.

If there is no x such that $f(x) = f(-x)$, then there exists an anti-pode preserving map from $S^2 \to S^1$. This is impossible. Proof: Let $g : S^2 \to S^1$ be an antipode preserving map. Then restricting S^2 to S^1, we have an anti-pode restricting map from S^1 to S^1, which can be extended to D^2 (as the map can be extended to S^2, and $D^2 \subset S^2$, we can easily extend the map to D^2). As the extension map from D^2 is nullhomotopic, we see that g is also nullhomotopic. However, we know that an antipode preserving map can never be nullhomotopic (the proof was beyond the scope of the document that I read). Hence, there has to exist x such that $f(x) = f(-x)$.

25. **Prove that $\pi_n(S^1) = 0$ for $n \geq 2$. What is $\pi_n(S^n)$?**

$\pi_n(S^1)$ is the set of homotopy equivalence classes of continuous maps from S^n to S^1. By the Lifting Theorem, any such map lifts uniquely to R. If we can prove that any map from S^n to R is contractible, then we're done.

Note that a continuous map $f : S^n \to R$ induces a map $f_* : \pi_1(S^n) \to \pi_n(R)$. As $\pi_n(R) = 0$, each map from S^n to R is contractible. Now observe that $f = f \circ id$, where $id : S^n \to S^n$ is the identity map. Hence we have the following maps $S^n \xrightarrow{id} S^n \xrightarrow{f} R$. Any from from S^n to R is homotopic to the constant map, as $\pi_n(R = 0$. Hence, $f \circ id$ is homotopic to the constant map. As $f \circ id = f$, we have that f is also homotopic to the identity map. Hence, the projection, which was the original map between S^n and S^1, is also nullhomotopic, proving that $\pi_n(S^1) = 0$ for $n \geq 2$.

$\pi_n(S^n) = Z$.

27. **Prove that S^n has a nowhere zero tangent vector field iff n is odd.**

Let us first assume that a nowhere zero tangent vector field exists. This implies that there is a map $S^n \to S^n$, defined in the following manner. First define a map $S^n \to \cup T(p)$, where $\cup T(p)$ is the union of tangent spaces for all $p \in S^n$, in the following manner: $p \to v(p)$. Here v is the non-zero vector field. Now define $p \to v(p) \to \frac{v(p)}{\|v(p)\|}$. This gives us a map between $S^n \to S^n$.

Now we know that any vector field is such that $v(p).p = 0$. Hence, $v(p)$ is orthogonal to both p and $-p$. This proves that we have a map from $S^n \to S^n$ such that no x is mapped to itself, which implies the degree of the map is $(-1)^{n+1}$, and no x is mapped to $-x$, which implies that the degree of the map is 1 (as the map can be shown to be homotopic to the identity map). Hence, as $(-1)^{n+1} = 1$ if and only if n is off, we have proven that n has to be odd for such a non-zero vector field to exist.

Now assuming that n is off, we shall prove that a non-zero vector field exists. Every point in S^n, where n is odd, can be written as an $n+1$-tuple. Note that $n+1$ is even. We can write $n+1$ as $2k$. Consider the following map: $v : (a_1, a'_1, a_2, a'_2, dots, a_k, a'_k) \to (a'_1, -a_1, a'_2, -a_2, \ldots, a'_k, -a_k)$. Clearly $v(p).p = 0$, which proves that this is a vector field. Moreover, it is non-zero as $\|v(p)\| = \|p\| \neq 0$. Hence proved.

28. **Prove that $H_p(CP^n) = Z$ when $p = 0, 2, 4, \ldots 2n$, and 0 otherwise.**

The CW complex of CP^n is $CP^0 \cup D^2 \cup D^4 \cup \cdots \cup D^{2n}$. Hence, there is one $2k$ cell for each $0 \leq k \leq n$, and no odd-dimensional cells. Using simplicial complexes, we have the following complex:

$$\cdots \to 0 \to C_{2n} \to 0 \to C_{2n-2} \to 0 \to \ldots$$

Hence, $H_{2k}(CP^n) = Z$ for $0 \leq k \leq n$, and 0 otherwise.

29. **State (do not prove) what are $H_p(RP^n)$ for $p = 0, 1, \ldots n$.**

The CW structure of RP^2 is formed by 1 copy each of $D_0, D_1, D^2, \ldots, D^n$. The boundary maps are these: $\partial(D^{2k-1}) = 0$ and $\partial(D^{2k}) = 2D^{2k-1}$. Hence, the CW homology of RP^n is the following:
$H_0(RP^n) = Z$
$H_{2k-1}(RP^k) = Z/2Z$
$H_{2k} = 0$
$H_n(RP^n) = Z$ if n is odd $H_n(RP^n) = 0$ if n is even

30. **Prove that if M is a C^∞ manifold, the space $C^r(M, R^n)$ i s smooth C^r function of M to R^n is dense in $C^0(M, R^n)$**

31. **Prove that a compact n-dim manifold admits an embedding in some R^k.**

This is just a reproduction of a part of the proof of the Whitney Embedding Theorem as given in Bredon.

Let us try and reproduce this argument.

Given a compact manifold M, let $\{U_i\}$ be an open cover. As M is compact, there exists a finite sub cover $\{U_i\}_{i=1}^n$. There also exists another finite cover $\{V_j\}_{j=1}^k$ such that each $V_j \subset \overline{V_j} \subset U_i$ for some i, and there exist functions λ_j such that the support of λ_j is contained within U_i, and $\lambda_j(x) = 1$ for $x \in V_j$.

Let ϕ_i be the charts for each U_i. Consider $\psi_j = \lambda_j(x)\phi_i(x)$ for $V_j \subset U_i$. There are also maps to R^n. Now consider the map $f : M \to \psi_1(x) \times \psi_2(x) \times \cdots \times \psi_k(x) \times \lambda_1(x) \times \cdots \times \lambda_k(x)$. We shall try to prove that this is an imbedding. We can do this in two steps: first prove that the map between tangent spaces is injective, and then prove that the map itself is injective.

First we prove that the map is injective. Let $f(p) = f(q)$. The fact that $\lambda_i(p)\lambda_i(q)$ tells us that they're in the same V_j. Now ψ_j is injective on V_j, as λ_j is 1 on each point of V_j. Hence, we have infectivity.

Now we prove that the map between tangent spaces is also injective.Each $x \in M$ is contained inside some V_j. As ψ_j is a homeomorphism on V_j, we know that $(\psi_j)_*(v)$ is non-zero for a non-zero tangent vector v. This proves that $f_* = (\psi_j)_* \times \cdots \times (\psi_k)_* \times (\lambda_1)_* \times \cdots \times (\lambda_k)_*$ is injective.

Hence, we've proved that M can be embedded in some R^k.

32. **Prove that given 2 C^∞ vector fields on a C^∞ manifolds, there exists a C^∞ vector field $[X, Y]$.**

$[X, Y]$ is defined as $XY - YX$. Let $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}$, where $\{x_1, \ldots, x_n\}$ are the local coordinates, then $XY - YX = \sum f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - g_i \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial x_j}$, which is a vector field. This is a C^∞ vector field as the coefficients, which are sums, products and derivatives of smooth functions derived from the coefficients of X and Y, are smooth. Hence proved.

33. **Show that the space X_M of C^∞ vector fields with the operators $+, [\ ]$ form a Lie algebra. Let X_M^c the sub(something) of vector fields of compact support. Let $[X_M^c, X_M^c]$ be the ideal formed by commutators. Show that $X_M^c = [X_M^c, X_M^c]$**

Vector fields are clearly closed under scalar multiplication and addition. Hence, they form a vector space. Moreover, they're also closed under bracket operation, as shown above. Hence, they form a Lie algebra.

Now we want to prove that $X_M^c = [X_m^c, X_M^c]$. First we shall show that $[X_M^c, X_M^c] \subset X_M^c$. Let X, Y be two vector fields with compact support. Then $XY - YX \in [X_M^c, X_M^c]$. Clearly, the support of $XY - YX$ is contained within the union of the supports of X and Y (as the supports of the derivatives is also contained within the supports of the functions). Hence, we see that every generator of $[X_M^c, X_M^c]$ belongs to X_M^c. Now it is also easy to observe that the sum of any two generators, or the product of any generator with a smooth function, will also have compact support. Hence, $[X_M^c, X_M^x] \subset X_M^c$.

Now we shall prove that $X_M^c \subset [X_M^c, X_M^c]$. For any $F \in X_M^c$, consider the vector field G that is 1 on the support of F, and also has compact support (although the support of the latter is clearly bigger than the support of f). Then $F = [G, F]$ This proves that $X_M^c \subset [X_M^c, X_M^c]$. Hence proved.

34. **Prove that $\pi_n(P^m)$ is trivial for $1 < n < m$.**

$\pi_n(P^m)$ is the set of homotopy equivalence classes of maps $f : S^n \to P^m$. By the Lifting Theorem, any such f is lifted to a unique $g : S^n \to S^m$, where $f = p \circ g$. Here p is the projection map from the universal covering space S^m to P^m.

This map g induces a map between $\pi_n(S^n)$ and $\pi_n(S^m)$. As $\pi_n(S^m) = 0$ for $m > n$, we have that g is contractible, which implies that $f = p \circ g$ is also contractible. Hence, $\pi_n(P^m) = 0$.